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Selective withdrawal and blocking wave in rotating fluids

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An analysis is made of the axially symmetric flow of a rotating inviscid incompressible fluid into a point sink at sufficiently low values of the Rossby number. Based on the experimental observations, a theoretical flow model involving a surface of velocity discontinuity which separates the central flowing core from the surrounding stagnant region is proposed. A family of solutions is obtained after posing the problem as one involving a free streamline which is the line of velocity discontinuity in the axial plane. It is found that the flow possesses a minimum flow force as well as a minimum energy flux. Corresponding to such a state, a unique intrinsic Rossby number R' based on the properties of the flowing core with a value of $1/\sqrt{8}$ is determined. A discussion is made of the flow field development induced by a sudden start of a sink discharge. A theoretical model involving a blocking wave propagating upstream is proposed. The speeds of blocking waves are found to be higher than the maximum group velocity of the infinitesimal waves for $R > 0{\cdot}06$. On the other hand, for $R < 0{\cdot}03$, the waves are linear and dispersive in nature.

1. Introduction

For the major part of this paper, an axially symmetric flow of a rotating inviscid incompressible fluid into a point sink at the axis of rotation, as shown in figure 1, will be considered. For values of the Rossby number R ($= Q/2\pi\Omega b^3$, where Q, b and Ω are the flux, radius and angular velocity of the rotating cylinder, respectively) greater than $0{\cdot}261$, a solution has been given by Long (1956). However, when R is near this critical value there is an elongated eddy extending axially upstream, resulting in return flow to infinity. The assumed upstream condition is hence violated and the flow pattern is also physically unstable, so that Long's solution does not describe the flow with R near the critical value. In the experimental investigation by Shih & Pao (1971, hereafter referred to as I), it has been demonstrated that, when R is near and below $0{\cdot}261$, the flow is essentially characterized by the presence of a stagnant region surrounding a core of flowing fluid. It was observed that at a Rossby number below the critical value the flow field,

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induced by a sudden start of discharge at the sink, experiences several distinct stages of development during the course of each experiment. During the second stage, the flowing core tends to adjust itself such that the intrinsic Rossby number R' ($= W_c/(2\Omega_c\delta_c)$, where δ_c, W_c and Ω_c are the radius, axial velocity and angular velocity of the flowing core, respectively) virtually remains at a constant value of 0·36 for all separated flows.

Theoretical works which show the features of blocking are recent papers by Trustrum (1964) and Kao (1965*a*, *b*). Trustrum developed a technique for solving initial-value problems using an Oseen-type approximation to the nonlinear inertial terms in the equations of motion. In particular, she solved the problem of a stratified flow into a line sink. A blocked flow was found. Kao constructed a free-streamline solution for stratified flows into a line sink and over obstacles at low Froude numbers.

In this paper, based on the experimental observations in I of the second stage of flow development, a theoretical flow model involving a surface of velocity discontinuity is constructed. The solution is, however, not unique and a family of solutions is found. Nevertheless, the non-uniqueness can be resolved by a plausible argument using the principle of minimum energy flux and flow force associated with the blocked flow.

In the last section the blocking wave associated with the establishment of a stagnant zone for the selective withdrawal is discussed. Since the experiments in I have established the overall character of the flow, a blocking flow model is now proposed. It is hoped that the present model may at least clear the ground for some subsequent attempt to provide a complete theory.

2. The governing equations

We shall consider axisymmetric motion only, and fixed cylindrical co-ordinates r, θ and z are used. With u, v and w denoting the velocity components in the directions of increasing r, θ and z, p^* denoting fluid pressure and χ denoting the body-force potential, the equations governing axisymmetric motion are

$$\frac{Du}{Dt}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.1}$$

$$\frac{Dv}{Dt}+\frac{uv}{r} = 0, \tag{2.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{2.3}$$

in which
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}, \quad p = p^*+\rho\chi.$$

The incompressibility condition may be satisfied by writing the velocity components u and w in the form

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{2.4}$$

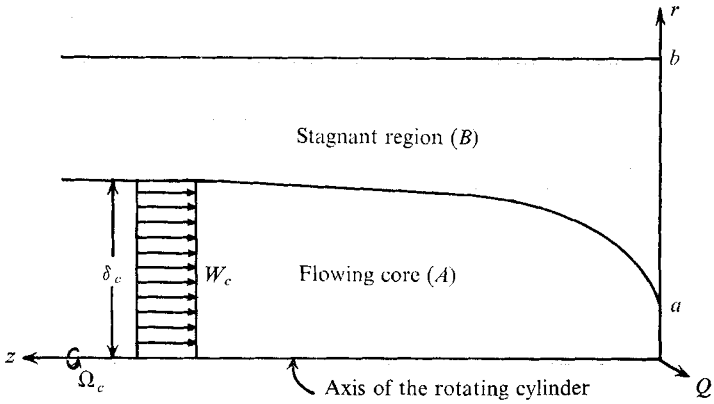

FIGURE 1. Blocking flow model in the axial plane.

where $\psi(r, z)$ is the stream function. Following Batchelor (1967, p. 545) equations (2.1)–(2.4) can be combined into one equation governing ψ:

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\psi = r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi}, \tag{2.5}$$

where $$H = \tfrac{1}{2}(u^2+v^2+w^2)+p/\rho, \quad C = vr, \tag{2.6}$$

which are functions of ψ only. This is essentially the azimuthal component of the vorticity equation.

Based upon the experimental observations we shall adopt a flow model as shown in figure 1. The model is characterized by a surface of velocity discontinuity. The fluid outside the discontinuity is stagnant in the axial plane and is, hence, not drawn into the sink. The dividing stream surface is a slipstream or a vortex sheet along which the pressure must be continuous. Now, in region A of figure 1, the fluid far upstream has a uniform axial velocity W_c and a uniform angular velocity Ω_c. Therefore, we have

$$\psi = -\tfrac{1}{2}r^2W_c, \quad C = r^2\Omega_c = -(2\Omega_c/W_c)\,\psi,$$

$$H = \tfrac{1}{2}W_c^2+r^2\Omega_c^2 = \tfrac{1}{2}W_c^2-(2\Omega_c^2/W_c)\,\psi.$$

The governing equation (2.5) for the flow in region A takes the linear form

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\psi = -\frac{2\Omega_c^2}{W_c}r^2-\frac{4\Omega_c^2}{W_c^2}\psi. \tag{2.7}$$

In dimensionless form, (2.7) becomes

$$\left(\frac{\partial^2}{\partial \eta^2}-\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{\partial^2}{\partial \xi^2}\right)\Psi = -\bar{R}^{-2}\Psi-\bar{R}^{-2}\eta^2, \tag{2.8}$$

where $\eta = r/b, \xi = z/b, \Psi = 2\psi/(b^2W_c)$ and $\bar{R} = W_c/(2\Omega_c b)$.

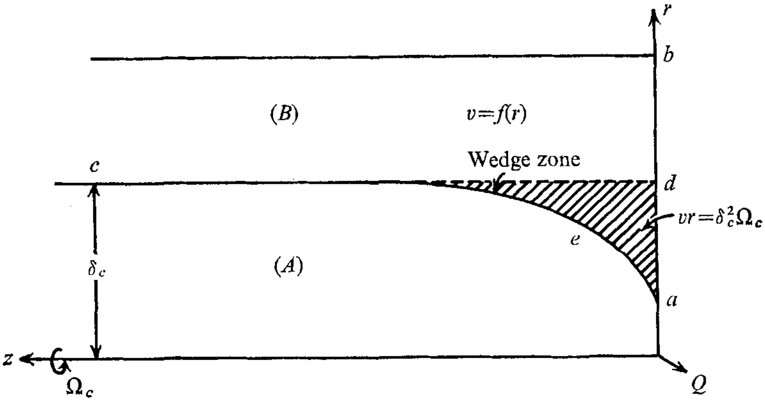

FIGURE 2. Wedge zone in the theoretical model.

3. The boundary conditions

With ξ measured axially from the sink and η measured radially from the axis, the boundary conditions in the flow region A are

$$\Psi = \begin{cases} 0 & \text{for} \quad \eta = 0, \;\; \xi > 0, \\ -\bar{\delta}_c^2 & \text{for} \quad 0 < \eta \leqslant \bar{a}, \;\; \xi = 0, \\ -\eta^2 & \text{for} \quad \xi = \infty, \end{cases} \tag{3.1}$$

where $\bar{\delta}_c = \delta_c/b$ is the dimensionless radius of the flowing core far upstream, and $\bar{a} = a/b$ is the radial distance to the point where the dividing streamline meets the line $\xi = 0$.

We consider the outer region B to be stagnant in the axial plane. The equation of motions are, therefore,

$$\frac{\partial p}{\partial r} = \rho \frac{v^2}{r}, \quad \frac{\partial p}{\partial z} = 0,$$

from which we obtain

$$v = f(r), \tag{3.2}$$

where $f(r)$ is an arbitrary function of r.

Now consider the wedge zone in region B as shown in figure 2. We assume that v is continuous across $r = \delta_c$ far upstream, so that (3.2) gives $rv = \delta_c^2 \Omega_c$ along line cd in region B. By virtue of (2.6) we also have $rv = \delta_c^2 \Omega_c$ along the dividing streamline cea. Thus, on the bounding surfaces of the wedge zone the angular momentum $rv = \delta_c^2 \Omega_c =$ constant. It follows that, unless the angular momentum vr is constant throughout the wedge zone, there must exist a layer of fluid in this wedge zone in which Rayleigh's stability criterion $d(rv)^2/dr > 0$ (Rayleigh 1880; Lin 1955, p. 51) is violated. As a consequence, mixing between the layers of different radii will result, making the angular momentum vr uniform throughout the wedge zone (cf. Scorer 1965; Bretherton & Turner 1968). It also follows that the swirling velocity v is continuous across the dividing streamline.

The dynamic boundary condition along the dividing streamline can be derived from Bernoulli's theorem. In region A along the dividing streamline we have, from (2.6),

$$\tfrac{1}{2}(u^2 + v^2 + w^2) + p/\rho = \text{constant}. \tag{3.3}$$

In the wedge region, $u = 0$, $w = 0$ and the swirling motion is a potential vortex; thus we have

$$\tfrac{1}{2}v^2 + p/\rho = \text{constant}. \tag{3.4}$$

Since the pressure p^*, the body-force potential χ and the swirling velocity v are continuous across the dividing streamline, on combining (3.3) and (3.4) we obtain

$$u^2 + w^2 = W_c^2$$

along the dividing streamline represented by $\psi = -\frac{1}{2}\delta_c^2 W_c$. Expressing this in non-dimensional form we have

$$\left(\frac{1}{2\eta}\frac{\partial \Psi}{\partial \xi}\right)^2 + \left(\frac{1}{2\eta}\frac{\partial \Psi}{\partial \eta}\right)^2 = 1. \tag{3.5}$$

4. Method of solution

The technique is to introduce a fictitious sink distribution from $\eta = \bar{a}$ to $\eta = 1$ at $\xi = 0$ in such a way that the governing equation (2.8) can be applied to the whole flow region. If the fictitious sink distribution ends at $\eta = \bar{a}$ the dividing streamline will be tangent to the line $\xi = 0$ at $\eta = \bar{a}$ and will divide the flow into two regions, one part flowing completely into the original sink and the other into the fictitious sink distribution that has been introduced. The fictitious sinks and their range $\bar{a}$ must be varied until the solution satisfies the dynamic condition (3.5) along the dividing streamline. This technique has been used successfully by Kao (1965*a*, 1970) in the problem of stratified flow into a line sink.

Let the solution of this new problem be Ψ. Using the principle of superposition we can assume that

$$\Psi = \gamma\Psi_1 + (1-\gamma)\Psi_2, \tag{4.1}$$

in which γ is the percentage of the total flow field that flows into the original sink, Ψ_1 represents flow into the original sink and Ψ_2 represents flow into the fictitious sink distribution.

The boundary conditions for Ψ_1 and Ψ_2 are

$$\Psi_1 = \begin{cases} 0 & \text{for} \quad \eta = 0, \quad 0 < \xi < \infty, \\ -\eta^2 & \text{for} \quad 0 \leqslant \eta \leqslant 1, \quad \xi = \infty, \\ -1 & \text{for} \quad 0 < \eta \leqslant 1, \quad \xi = 0, \\ -1 & \text{for} \quad \eta = 1, \quad 0 < \xi < \infty \end{cases} \tag{4.2}$$

and

$$\Psi_2 = \begin{cases} 0 & \text{for} \quad \eta = 0, \quad 0 < \xi < \infty, \\ 0 & \text{for} \quad 0 < \eta \leqslant \bar{a}, \quad \xi = 0, \\ -\eta^2 & \text{for} \quad 0 \leqslant \eta \leqslant 1, \quad \xi = \infty, \\ g(\eta) & \text{for} \quad \bar{a} \leqslant \eta \leqslant 1, \quad \xi = 0, \\ -1 & \text{for} \quad \eta = 1, \quad 0 < \xi < \infty, \end{cases} \tag{4.3}$$

where $g(\eta)$ is the distribution function for the fictitious sink. The dividing streamline is represented by

$$\Psi = -\gamma = -\bar{\delta}_c^2. \tag{4.4}$$

The solution Ψ satisfying all boundary conditions listed in (4.2) and (4.3) and the dynamic boundary condition (3.5) will exhibit the dividing streamline of the original boundary-value problem. The region that flows into the fictitious sink distribution is then replaced by the stagnant fluid. Solutions for Ψ_1 and Ψ_2 can be obtained by the method of separation of variables. For Ψ_1 we obtain

$$\Psi_1 = -\eta^2 + \eta \sum_{n=1}^{\infty} A_n \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta), \tag{4.5}$$

in which $A_n = -2/[\lambda_n J_0^2(\lambda_n)]$ and λ_n is the nth zero of $J_1(\lambda)$. For Ψ_2, a constant sink strength from $\eta = \bar{a}$ to $\eta = 1$ is assumed.† Thus we have

$$g(\eta) = (\bar{a}^2 - \eta^2)/(1-\bar{a}^2). \tag{4.6}$$

With $g(\eta)$ so chosen, it is then found that

$$\Psi_2 = -\eta^2 + \eta \sum_{n=1}^{\infty} B_n \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta), \tag{4.7}$$

in which
$$B_n = \frac{4\bar{a}}{1-\bar{a}^2}\,\frac{J_1(\bar{a}\lambda_n)}{\lambda_n J_0^2(\lambda_n)}.$$

Finally we obtain

$$\Psi = -\eta^2 + \gamma\eta \sum_{n=1}^{\infty} A_n \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta) + (1-\gamma)\,\eta \sum_{n=1}^{\infty} B_n \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta). \tag{4.8}$$

The series converges uniformly for all values of $\xi > 0$ and $0 \leqslant \eta \leqslant 1$. The dimensionless velocity components are

$$-\frac{1}{2\eta}\frac{\partial \Psi}{\partial \xi} = \frac{\gamma}{2} \sum_{n=1}^{\infty} A_n' \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta) + \frac{1-\gamma}{2} \sum_{n=1}^{\infty} B_n' \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_1(\lambda_n \eta), \tag{4.9}$$

where
$$A_n' = -\frac{2(\lambda_n^2 - R^{-2})^{\frac{1}{2}}}{\lambda_n J_0^2(\lambda_n)}, \quad B_n' = \frac{4\bar{a}}{1-\bar{a}^2}\,\frac{J_1(\bar{a}\lambda_n)\,(\lambda_n^2 - R^{-2})^{\frac{1}{2}}}{\lambda_n J_0^2(\lambda_n)}$$

and

$$\frac{1}{2\eta}\frac{\partial \Psi}{\partial \eta} = -1 + \frac{\gamma}{2} \sum_{n=1}^{\infty} A_n'' \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_0(\lambda_n \eta) + \frac{1-\gamma}{2} \sum_{n=1}^{\infty} B_n'' \exp\left[-(\lambda_n^2 - \bar{R}^{-2})^{\frac{1}{2}}\xi\right] J_0(\lambda_n \eta), \tag{4.10}$$

where
$$A_n'' = \frac{-2}{J_0^2(\lambda_n)}, \quad B_n'' = \frac{4\bar{a}}{1-\bar{a}^2}\,\frac{J_1(\bar{a}\lambda_n)}{J_0^2(\lambda_n)}.$$

† One of the referees has pointed out that the uniform sink distribution $\bar{a} < \eta < 1$ and the isolated sink at $\eta = 0$ will produce a stagnation point at some point between 0 and $\bar{a}$. This is true because there is a local singularity of a logarithmic nature for the radial velocity component at $\eta = \bar{a}$, if the strength of the sink distribution is finite there. However, the logarithmic singularity is felt only in a small neighbourhood of the point $\eta = \bar{a}$, so that, by letting the uniform sink distribution to fall off to zero in a small interval near $\eta = \bar{a}$, the singularity is easily removed and tangency at $\eta = \bar{a}$ is assured for a sufficiently strong sink at the origin. Our colleague, Professor T. W. Kao, has kindly provided a proof of the above point, which is now included as an appendix here.

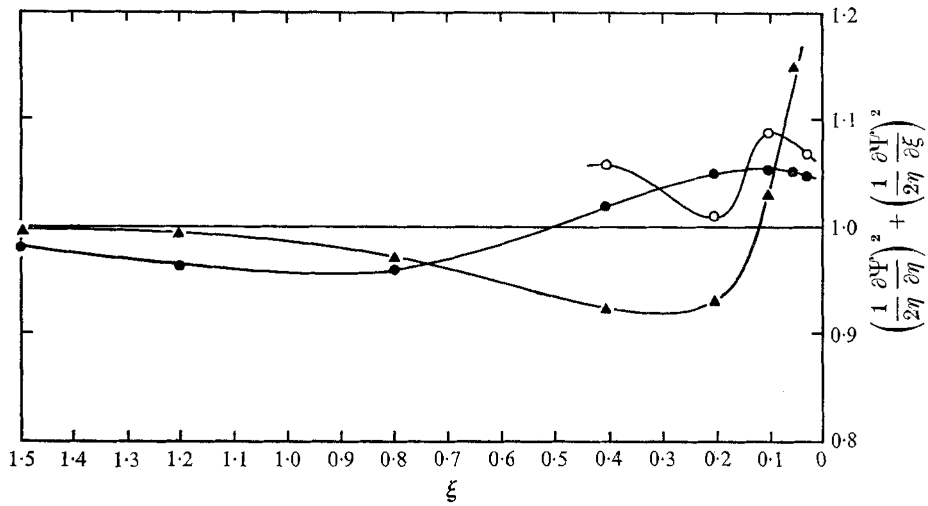

FIGURE 3. Deviation from the dynamic boundary condition for some computed solutions.

	○	●	▲
$\bar{R}$	0·27	0·50	0·50
γ	0·51	0·50	0·49
$\bar{a}$	0·38	0·31	0·35

Since there are three parameters, γ, $\bar{a}$ and $\bar{R}$, involved in these expressions, a trial-and-error method is used to solve the problem. For any assumed values of γ, $\bar{a}$ and $\bar{R}$ we first use (4.4) and (4.8) to determine the correct position of the dividing streamline $\eta = \eta(\xi)$. Equations (4.9) and (4.10) then enable us to calculate the velocity along the dividing streamline. It is seen that if a graph of dimensionless speed against ξ is plotted it can immediately be determined whether the dynamic boundary condition (3.5) along the dividing streamline is satisfied. Some typical results are presented in figure 3. It is found that the dynamic boundary condition (3.5) along the dividing streamline is only satisfied approximately but the errors involved are small. Moreover, from the numerical result, it is found that stream functions and positions of the dividing streamline are not very sensitive to small errors in the dynamic boundary condition (3.5).† Detailed calculations were done with the aid of the CEIR computing service and an IBM 7094 computer.

5. Flow pattern

The flow pattern of a typical solution is shown in figure 4. The solution below the dividing streamline is the solution to the originally posed boundary-value problem with intrinsic Rossby number $R' = \bar{R}/\gamma^{\frac{1}{2}}$, where $R' = W_c/(2\Omega_c \delta_c)$.

The range of R' for the solutions calculated is from 0·36 to 13·57. The flow pattern for a large Rossby number is plotted in figure 5. Since for large Rossby

† A similar situation was also found by Kao (1970) in his calculation of stratified flow. In fact Kao has demonstrated this very convincingly by comparing his calculated solution with a potential solution.

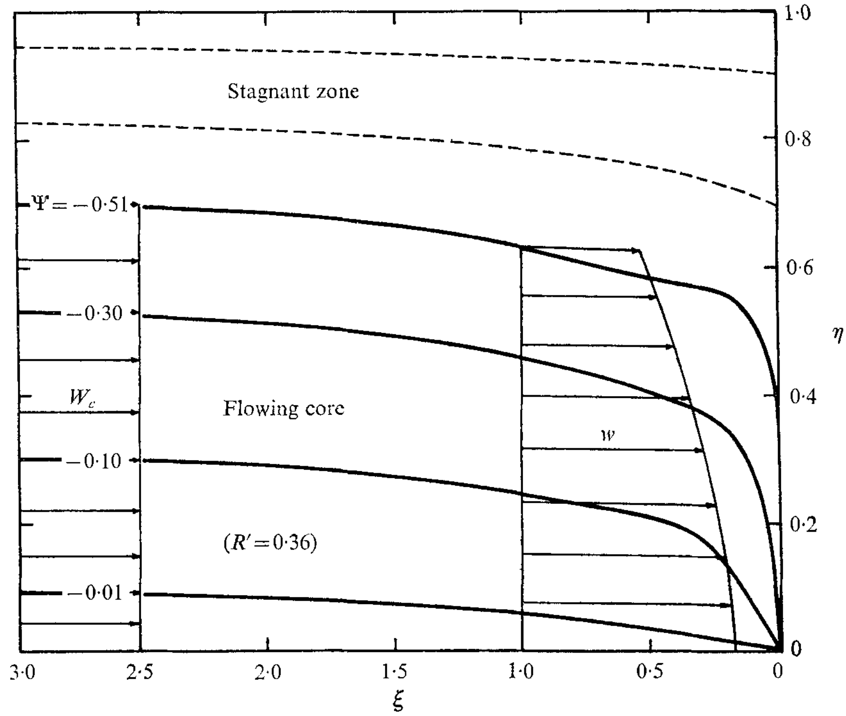

FIGURE 4. Theoretical flow of selective withdrawal toward a point sink on the axis of rotation. The streamlines and the velocity profiles are in the axial plane.

number the effect of rotation is negligible, the flow pattern in figure 5 can, therefore, be interpreted as the flow discharging into the air from a hole in the bottom of a large tank containing liquid. The dividing stream surface is then a free surface. The gravity effect is, however, being neglected here. It is noted that such a flow pattern must exist at least near the point source where the velocity is large and the gravity effect is indeed negligible.

It becomes apparent that there exists a family of solutions corresponding to a range of values of R'. Like most inviscid fluid problems with singularities in the flow field, the non-uniqueness of the solution is a direct consequence of the singularities involved. In the present case the dividing streamline, which is a vortex sheet, is certainly a sheet of singularity.

Kao (1970), investigating the problem of the flow of a stratified fluid into a line sink, has recently found that the free-streamline solution is indeed not unique and has given a heuristic argument for the choice of the intrinsic Froude number for selective withdrawal. His argument gave a unique Froude number $F' = 0{\cdot}33$ which is slightly higher than π^{-1}. Because of the strong similarities between his problem and the present one for rotating fluids it is expected that by a similar approach a unique solution, corresponding to a unique intrinsic Rossby number which is slightly higher than 0·261, will be determined. No such attempt is made at the present time. In spite of the fact that Kao's argument is self-consistent within the framework of steady inviscid flows, nevertheless, the unique number, derived from his argument, seems not definitive.

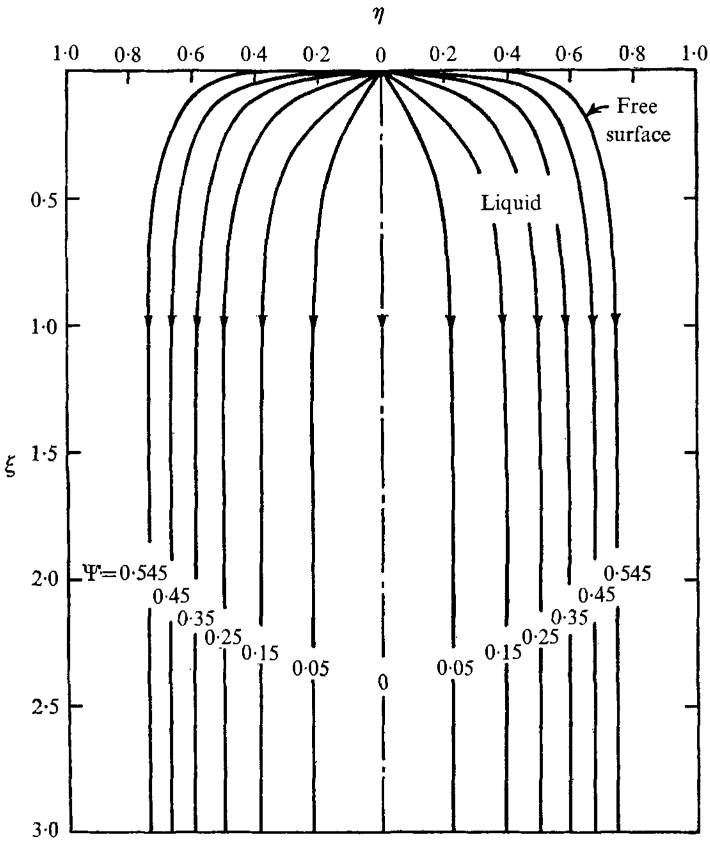

FIGURE 5. Theoretical flow of discharge from a point source for a large Rossby number, $R' = 13{\cdot}57$, $\gamma = 0{\cdot}545$, $\bar{a} = 0{\cdot}39$.

In the present case of rotating flows, the unique number derived from the experimental study in I is $R' = 0{\cdot}36$, which is quite a bit higher than the value expected from Kao's argument. Moreover, Long's solution (1956) shows that the effective radius of the flowing core and its rotational speed undergo a drastic change as the flow approaches the critical Rossby number 0·261. It is, therefore, quite natural that the value of the intrinsic Rossby number R' of the flowing core may differ considerably from the value of the critical Rossby number at which the flow begins to separate. Although a complete theory which would predict the unique value of the intrinsic Rossby number of all separated flows is still lacking, a consideration of the energy flux and flow force in the following section will bring some physical significance to the unique intrinsic Rossby number 0·36 obtained from the experimental investigation.

6. Energy flux and flow force

Energy flux

From the experimental evidence in I, a quasi-steady inviscid core flow of selective withdrawal does exist as shown in figure 1. The total energy transport through a given section is

$$E = \int_0^{2\pi} \int_0^{\infty} \rho H w r \, dr \, d\theta, \tag{6.1}$$

where H is the Bernoulli quantity defined in (2.6) and the integration is to be performed over any cross-sectional area. Since w is zero outside the flowing core A, it is therefore sufficient to integrate over the cross-sectional area of the flowing zone. Thus we have

$$E = 2\pi\rho \int_0^{\delta} H \frac{\partial \psi}{\partial r} dr = 2\pi\rho \int_0^{\psi_a} H(\psi)\, d\psi = \text{constant} \quad \text{along the flow.}$$

Evaluating the energy flux E upstream we have

$$E = 2\pi\rho \int_0^{\delta_c} \left(\tfrac{1}{2} W_c^2 + \Omega_c^2 r^2 + \frac{p_0}{\rho}\right) W_c r \, dr = \rho Q \left(\frac{Q^2}{2\pi^2 \delta_c^4} + \tfrac{1}{2}\,\Omega_c^2 \delta_c^2 + \frac{p_0}{\rho}\right), \tag{6.2}$$

where p_0 is the value of p at the axis of rotation and $Q = \pi \delta_c^2 W_c$ is the volume flux through the sink. It is seen that, for given values of Q and Ω_c, the energy flux is a function of the radius δ_c of the flowing core. For given Q and Ω_c, any radius for the flowing core is possible but, for a certain radius b_c, the energy flux is a minimum. The minimum occurs when $\partial E/\partial \delta_c = 0$. On using (6.2) with p_0 being independent of δ_c this yields

$$b_c^3 = \frac{Q\sqrt{2}}{\pi \Omega_c} \quad \text{or} \quad \frac{Q}{2\pi \Omega_c b_c^3} = \frac{1}{2\sqrt{2}}. \tag{6.3}$$

The second relation in (6.3) is equivalent to $R' = 0{\cdot}354$. This value of R' is quite close to the experimental value 0·36 in I. This means that out of a whole class of possible flowing cores, for given values of Q and Ω_c, only one radius will give a minimum energy flux. The fact that (6.3) agrees with the experimental observation is significant since this verifies the very important principle that whenever selective withdrawal occurs the flowing core tends to adjust itself such that the flow possesses a minimum energy flux. Such a flow will now conveniently be called a pseudo-critical blocked flow since it is not a critical flow in the usual sense.

Flow force

We define

$$S = \int_0^{2\pi} \int_0^{\delta} (p + \rho w^2)\, r \, dr \, d\theta, \tag{6.4}$$

where S is called flow force, being the sum of axial pressure force and momentum flux. Since the flow force is constant in the present flow it can be evaluated upstream. At the upstream section we have

$$S = 2\pi\rho \int_0^{\delta_c} \left(W_c^2 + \tfrac{1}{2} r^2 \Omega_c^2 + \frac{p_0}{\rho}\right) r \, dr. \tag{6.5}$$

Setting the reference pressure p_0 at the axis equal to zero and integrating (6.5) we obtain

$$S = \pi\rho(Q^2/\pi^2\delta_c^2 + \tfrac{1}{4}\delta_c^4\Omega_c^2). \tag{6.6}$$

For given values of Q and Ω_c, the flow force is a minimum for a certain radius b_c of the flowing core. The minimum occurs when $\partial S/\partial\delta_c = 0$; this yields

$$b_c^3 = \frac{Q\sqrt{2}}{\pi\Omega_c} \quad \text{or} \quad \frac{Q}{2\pi\Omega_c b_c^3} = \frac{1}{2\sqrt{2}}. \tag{6.7}$$

Relation (6.7) is exactly the same as (6.3). Therefore, we conclude here that, whenever selective withdrawal in a rotating fluid occurs, the flowing core tends to adjust itself such that the flow possesses a minimum flow force as well as a minimum energy flux.

Maximum discharge rate

There is another way of looking at the energy flux and the flow force. From (6.2) for given values of E and Ω_c, the discharge rate Q is a function of the radius δ_c of the flowing core. For given E and Ω_c, any radius of flowing core is possible but, for a certain radius b_c, the discharge rate is a maximum. The maximum occurs when $\partial Q/\partial\delta_c = 0$. On using (6.2) with p_0 remaining constant, this yields

$$b_c^3 = Q\sqrt{2}/\pi\Omega_c.$$

This is exactly the same as (6.3) or (6.7). Since (6.3) has been verified by the experimental study in I, it follows that the flowing core tends to adjust itself such that the flow possesses a maximum discharge rate for given values of E and Ω_c.

Similary, from (6.6) it is seen that, for given values of S and Ω_c, any radius of flowing core is possible but, for a certain radius b_c, the discharge rate is a maximum. The maximum occurs when $\partial Q/\partial\delta_c = 0$; this yields, from (6.6),

$$b_c^3 = Q\sqrt{2}/\pi\Omega_c.$$

This again means that Q will take on its maximum value for given S and Ω_c when (6.3) holds. It then follows that the flowing core tends to adjust itself such that the flow possesses a maximum discharge rate for given values of S and Ω_c.

7. Blocking wave

The experimental observation in I actually carried out an initial-value problem for a sink flow in a rotating fluid. It was observed experimentally that at a Rossby number below the critical value the flow field, induced by a sudden start of discharge at the sink, experiences several distinct stages during the course of each run. At the initial moment the flow exhibits a feature of potential flow. It then develops into a state of selective withdrawal with an inviscid profile of a flowing core surrounded by an almost stagnant region. The flow reaches a quasi-steady state during this stage. An equivalent initial-value problem can now be stated. The flow initially consists of a unidirectional flow in the negative-z direction with uniform velocity W and uniform angular velocity Ω, inside a circular cylindrical tube of uniform radius. At the time $t = 0$ a disk with a small hole in its centre is inserted perpendicular to the flow such that the flow is blocked in the entire section of the tube except at the central hole, through which the fluid is discharged like a sink flow; the strength of the sink is being maintained at a constant value. It is known that a potential flow will be established immediately (see Lamb 1932, p. 11). In the meantime, it is expected that a large disturbance will be propagated upstream owing to this blockage if the original flow is subcritical† (i.e. $W/(2\Omega b) < 0{\cdot}261$). An attempt to solve such an initial-value problem has been made by Trustrum (1964) for a density-stratified flow, using an Oseen-type approximation to the nonlinear inertial terms in the equations of motion. For the problem of a stratified flow into a line sink, a blocked flow was found. Although Trustrum's large-time solutions seem to agree qualitatively with the observations made by Long (1956) and Debler (1959), her solution has some shortcomings.

Trustrum's solution is to be questioned because of her Oseen approximation. For the Oseen approximation to be valid, the velocity perturbation must be much smaller than the free-stream velocity. Nevertheless, when blocking occurs, the Oseen approximation is violated in her solution (i) near the sink or obstacle all the time, and (ii) far upstream as a steady state is approached.

In the experimental investigation in I, some fluctuations of the flow field in the initial stage have been detected, although no measurement has been made of the details of the wave motion. From the experiment of sudden discharge at the sink in I, a quasi-steady flowing core surrounded by a stagnant zone was established at $z = 9$ in. about 5–50 s after the sink discharge, depending on values of Q and Ω. Since disturbances thus generated travel at finite speeds, the flow far upstream must still remain undisturbed. A sketch depicting the development of the blocking wave is shown in figure 6 (*b*). Although the detailed structure of the blocking wave is yet to be studied both experimentally and theoretically, the existence of such a wave seems beyond any doubt. Since the wave motions in a rotating fluid contained in a tube, density-stratified fluid flowing between horizontal boundaries, and water in an open channel have strong similarities (see Long 1954; Benjamin 1970; Veronis 1970) some phenomena in one physical system can be analysed and understood in terms of the behaviour in the analogous system. To understand the phenomenon of blocking waves in a rotating fluid it is advisable to note here that analogous phenomena have been observed by Long (1954, 1970) and previous investigators in open-channel and two-layer fluid flow. The analogy is, in fact, quite striking. The following paragraph is essentially taken from Long's (1954) paper concerning the hydraulic analogy.

When a high obstacle is inserted in an initially subcritical open-channel flow, the flow will become supercritical momentarily at the crest, although remaining subcritical upstream and downstream. A steady state cannot be maintained, of course, and a wave of elevation (which may break forming an upstream bore) will progress upstream raising the upstream level sufficiently to permit a critical condition at the crest. A steady state will ultimately be established with a lee jump.

From the above description, it is seen that this wave of elevation propagating

† In a very recent paper, Long (1970) pointed out that in a two-fluid system the fluid upstream may be disturbed even when conditions permit no infinitesimal disturbance to progress upstream.

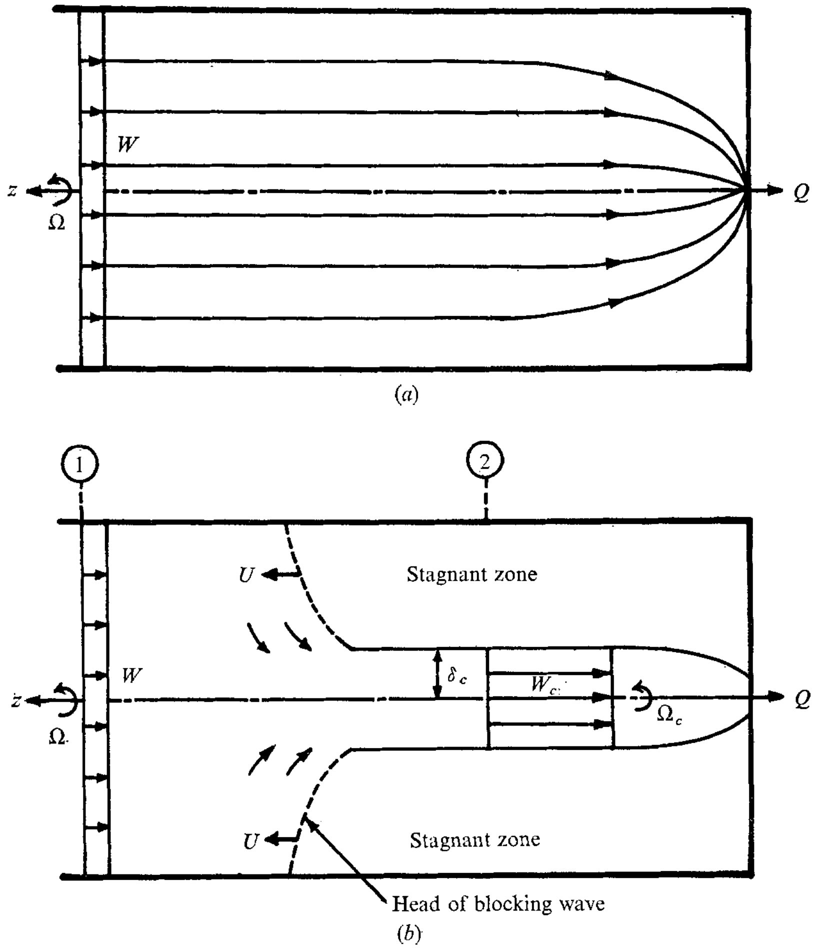

FIGURE 6. Development of the blocking wave and stagnant zone. (*a*) $t = 0+$, potential flow into a point sink. (*b*) At a later time, the blocking wave progressing upstream with speed U, leaving a stagnant zone and a concentrated flow behind it.

upstream is analogous to the blocking wave shown in figure 6. A similar wave of elevation had been observed by Long (1954) in a two-layer flow system. In rotating or continuously stratified fluids, development of a vortex sheet is permissible, and blocking is manifested in the form of stagnant zones in the flow field. Thus, instead of raising the upstream level as in open-channel and two-fluid flows, the front of the stagnant zone will progress upstream. Behind this blocking wave, an almost steady pseudo-critical blocked flow as discussed in the previous section will be established. Thus, after a sufficient time, a virtually steady flow of selective withdrawal will be established as was observed experimentally in I. It is seen that the flow upstream of the blocking wave is still in the first stage of flow development (potential flow) as classified by the experimental study in I while the flow behind the blocking wave should properly belong to the second stage. Our aim here is to establish certain relationships between the speed of this blocking wave packet and the parameters of the flow field and to explain certain phenomena observed in the development of selective withdrawal in I.

In the following analysis it is assumed that the flow far downstream of the blocking wave, as shown at cross-section 2 in figure 6 (b), is steady with uniform axial velocity W_c and uniform angular velocity Ω_c in the central core region of radius δ_c and zero axial velocity surrounding the core. The above assumption was essentially verified by the experimental results in I.

Now, we define
$$M = \rho \int_0^{2\pi} \int_0^b wr\,dr\,d\theta,$$
where M is the mass flux through any cross-section. Since obviously the disk cannot displace the fluid at $z = +\infty$ when only a finite time has elapsed from the start of the motion, conservation of mass requires M to be constant. Now consider two cross-sections, one far upstream and the other far downstream of the blocking wave as shown in figure 6 (b). Thus, between sections 1 and 2 we have
$$Wb^2 = W_c \delta_c^2 = Q/\pi. \tag{7.1}$$
For $z > 0$ the equation expressing conservation of momentum is
$$\partial M/\partial t + \partial S/\partial x = 0, \tag{7.2}$$
where, as before, S is the flow force defined by
$$S = \int_0^{2\pi} \int_0^b (p+\rho w^2)\, r\,dr\,d\theta. \tag{7.3}$$
Since M is constant, (7.2) shows that the value of the flow force S must be constant everywhere for $z > 0$. Thus, we obtain between sections 1 and 2
$$\int_0^b (p_1 + \rho W^2)\, r\,dr = \int_0^{\delta_c} (p_2 + \rho W_c^2)\, r\,dr + \int_\delta^b p_2 r\,dr = S/2\pi, \tag{7.4}$$
or, after using (7.1),
$$\int_0^b (p_1 - p_2)\, r\,dr = \rho Q(W_c - W)/2\pi. \tag{7.5}$$

We now suppose that the blocking wave packet has been progressing upstream for some time and that its speed† has reached a constant value U. Viewed from a frame of reference moving with the blocking wave, the flow situation in 6 (b) is as shown in figure 7. The upstream flow is then maintained with a uniform velocity $U + W$, while the downstream flow consists of a concentrated flow at the central portion with a uniform velocity $U + W$ and a uniform flow with a velocity U surrounding the flowing core. The blocking wave (transition zone) now appears stationary in this moving frame of reference, although some minor unsteady fluctuations may exist there in actual flows. Whether the blocking wave will break as it progresses upstream is yet to be investigated, although some evidence of weak turbulence was indeed observed during the initial stage of development in

† The exact structure and behaviour of the blocking wave packet are yet to be investigated. The wave packet is probably dispersive in nature; therefore, its speed is the average value for the wave group.

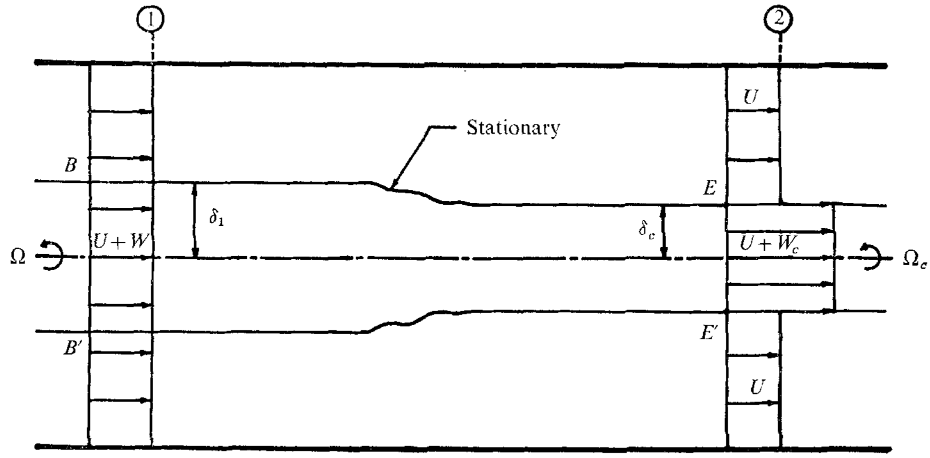

FIGURE 7. Rotating flow relative to a co-ordinate system moving with the blocking wave.

the experiments. As it turns out, some energy dissipation is indeed essential in the transition region. This point will be discussed later in the section.

From either figure 6 (*b*) or figure 7 it is seen that radial convergence of the flow field is induced as the blocking wave progresses. This leads to convergence of the angular momentum, which is the reason why the angular velocities in the flowing core during the second stage of selective withdrawal were observed to be substantially higher than the basic rotation. Since the time required to establish the second stage was quite short (5–50 s) and the flow is essentially inviscid in nature during this stage, it is seen that the 'spin-up' of the central core is almost entirely due to the convergence of the angular momentum. In this respect it is quite similar to the spin-up in a closed container (see Greenspan 1968, p. 35).

Now by considering the angular momentum in a control volume coinciding with the stream tube BB'–EE' (see figure 7), the angular momentum flux at sections 1 and 2 must be equal owing to conservation of the angular momentum flux; thus we have

$$\int_0^{\delta_1} \Omega r^2 (U+W)\, r\, dr = \int_0^{\delta_c} \Omega_c r^2 (U+W_c)\, r\, dr. \tag{7.6}$$

Since, from continuity along the stream tube, we have

$$(U+W)\,\delta_1^2 = (U+W_c)\,\delta_c^2, \tag{7.7}$$

equation (7.6) then becomes

$$\Omega\delta_1^2 = \Omega_c \delta_c^2. \tag{7.8}$$

This means that the stream tube BB'–EE' is essentially a vortex tube and that the strength along the vortex tube is constant.

Now we want to show why the angular velocity Ω_c at the central portion of section 2 is uniform if the axial velocity W_c is uniform there. Let r_1 and r_2 be the radii of a stream tube in sections 1 and 2. From continuity along the stream tube, we have

$$(U+W)\,r_1^2 = (U+W_c)\,r_2^2 \quad \text{for} \quad 0 \leqslant r_2 \leqslant \delta_c, \tag{7.9a}$$

$$(U+W)\,(r_1^2-\delta_1^2) = U(r_2^2-\delta_c^2) \quad \text{for} \quad \delta_c \leqslant r_2 \leqslant b. \tag{7.9b}$$

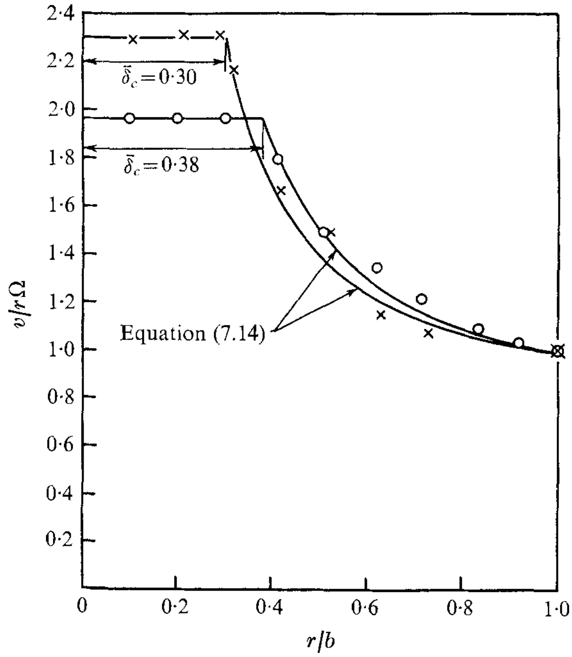

FIGURE 8. Typical tangential velocity profiles: ×, experimental data points for $\Omega_c/\Omega = 2{\cdot}30$, $R = 0{\cdot}037$; ○, experimental data points for $\Omega_c/\Omega = 1{\cdot}97$, $R = 0{\cdot}054$.

We now assume that the angular momentum along the stream tube is conserved; thus we have

$$v_2 r_2 = v_1 r_1 = \Omega r_1^2, \tag{7.10}$$

where v_1 and v_2 are tangential velocities at sections 1 and 2, respectively. By virtue of (7.9*a*), equation (7.10) becomes

$$v_2/r_2 = \Omega(U+W_c)/(U+W) \quad \text{for} \quad 0 \leqslant r_2 \leqslant \delta_c, \tag{7.11}$$

which yields on account of (7.7)

$$v_2/r_2 = \Omega\delta_1^2/\delta_c^2. \tag{7.12}$$

Therefore, the angular velocity at the central portion of section 2 is indeed uniform. To get the tangential velocity in the *outer region* $\delta_c \leqslant r_2 \leqslant b$, we substitute (7.9*b*) into (7.10); thus we obtain

$$\frac{v_2}{r_2\Omega} = \frac{1}{\bar{r}_2^2}\left[\frac{1-\bar{\delta}_1^2}{1-\bar{\delta}_c^2}(\bar{r}_2^2-\bar{\delta}_c^2)+\bar{\delta}_1^2\right], \tag{7.13}$$

where $\bar{r}_2 = r_2/b$, $\bar{\delta}_1 = \delta_1/b$ and $\bar{\delta}_c = \delta_c/b$. In view of (7.8), equation (7.13) can also be expressed as

$$\frac{v_2}{r_2\Omega} = \frac{1}{\bar{r}_2^2}\left[\frac{1-(\Omega_c/\Omega)\,\bar{\delta}_c^2}{1-\bar{\delta}_c^2}(\bar{r}_2^2-\bar{\delta}_c^2)+\frac{\Omega_c}{\Omega}\bar{\delta}_c^2\right]. \tag{7.14}$$

This result is independent of U. Thus (7.14) should still hold even if the transition region is unsteady. Equation (7.14) is now compared with the experimental results. Two typical tangential velocity profiles from the experimental measurements are shown in figure 8. With the ratio Ω_c/Ω known from the experiments and

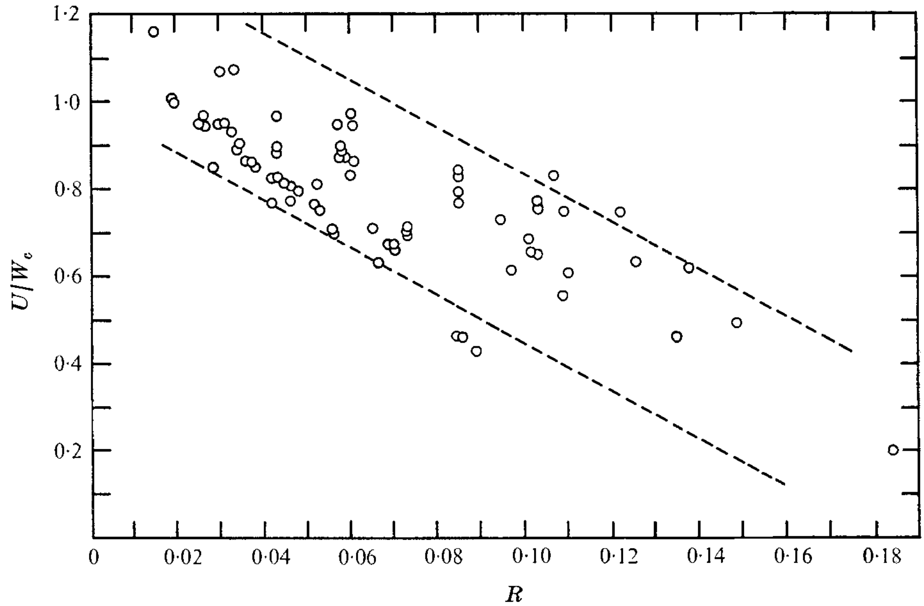

FIGURE 9. Advancing speed of the blocking wave *vs.* Rossby number.

$\bar{\delta}_c$ determined from the experimental data, the tangential velocities in expression (7.14) are also plotted in figure 8. It is seen that the agreement between the experimental data points and equation (7.14) is remarkably good. This verifies indirectly that the flow model proposed in figure 7 is probably a valid one, and that the assumptions made above are appropriate. It is also noted that the tangential velocity in figure 8 is different from the Rankine vortex since the outer portion is not a potential vortex.

Now we proceed to find the speed of advancement of the blocking wave packet. From (7.7) and (7.8) we obtain

$$\Omega_c/\Omega = (U+W_c)/(U+W). \tag{7.15}$$

On using (7.1), equation (7.15) becomes

$$\Omega_c/\Omega = (\bar{U}+1)/(\bar{U}+\bar{\delta}_c^2), \tag{7.16}$$

where $\bar{U} = U/W_c$. We observe that Ω_c/Ω is always greater than one since $\bar{\delta}_c^2 < 1$ (in fact $\bar{\delta}_c^2$ is usually much smaller than one). Since Ω_c/Ω and $\bar{\delta}_c^2$ were measured in the experiment, the values of $\bar{U}$ can thus be determined from (7.16). In the experiment, the value of Ω_c/Ω ranges from 1·746 to 2·843 while $\bar{\delta}_c$ varies from 0·256 to 0·606. The values of $\bar{U}$ determined from equation (7.16) are now plotted in figure 9. It is seen that the value of U/W_c decreases as R increases, where the broken lines in the figure indicate the general trend. With W_c known from the experiment, the values of U can then be determined. To compare U with the maximum group velocity c_0 (= 0·522Ωb) of the infinitesimal waves in a rotating tube, the ratio U/c_0 is now plotted in figure 10 (see Fraenkel 1956). It is seen that the majority of points are centred around $U/c_0 = 0·8$. If we now use $U \simeq 0·8c_0$ (= 0·418Ωb)

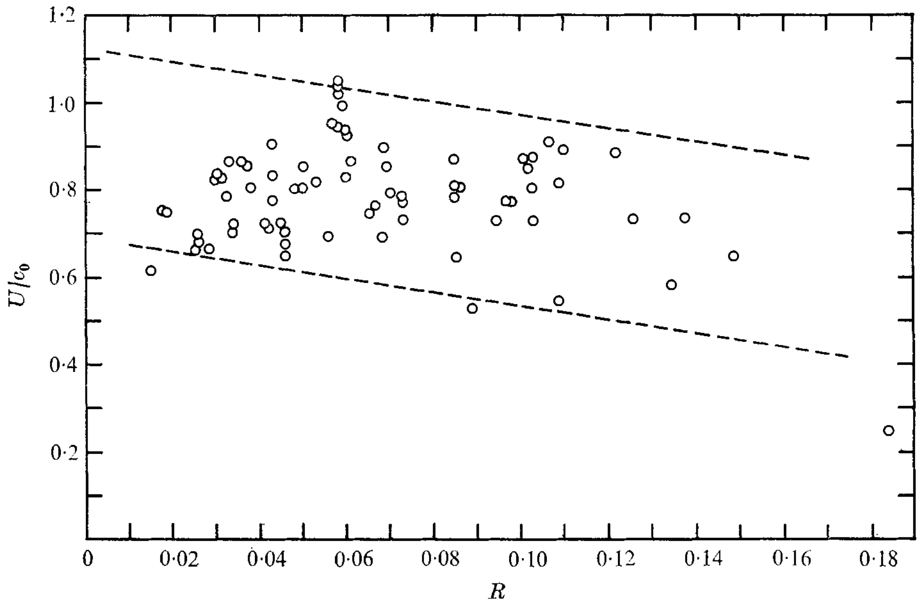

FIGURE 10. Ratio of the advancing speed of the blocking wave and the maximum group velocity of the infinitesimal waves *vs.* Rossby number.

for the advancing speed of the blocking wave, then the time required for the blocking wave to travel a distance of 9 in. can easily be estimated. We obtained $t_0 = 4{\cdot}42/\Omega$ (in seconds). Thus, for $\Omega = 0{\cdot}079$ rad/s we have $t_0 = 55{\cdot}9$ s and for $\Omega = 0{\cdot}817$ rad/s we have $t_0 = 5{\cdot}41$ s. It was observed in the experiment that the time required to establish the second stage at $z = 9$ in. was about 5 s for $\Omega = 0{\cdot}817$ rad/s and 50 s for $\Omega = 0{\cdot}079$ rad/s. The fact that the time required to establish the second stage at $z = 9$ in. is so close to the time for the blocking wave to travel such a distance indicates that the quasi-steady second stage at a certain position is now established as soon as the blocking wave has passed through it.

We now note from figure 7 that the blocking wave packet is stationary in a uniform upstream flow with velocity $U + W$. Therefore, the wave speed c relative to the medium should be $U + W$. Although W is small compared with U, in the majority of experiments the correction is significant. Since W was measured very accurately in the experiment, the values of c are now calculated and the ratio c/c_0 is plotted in figure 11. It is seen that the majority of c/c_0 values are substantially greater than unity for $R > 0{\cdot}06$. From the hydraulic analogy with the open-channel flow (Long 1954, 1970), this indicates that the blocking wave is nonlinear in nature and that the finite disturbance progresses upstream at a supercritical speed. For $R < 0{\cdot}03$, it is seen that the values of c/c_0 are substantially less than unity. This seems to indicate that the behaviour of the wave for $R < 0{\cdot}03$ is essentially linear in nature. A recent investigation (Pao & Kao 1972) of a point sink flow in a rotating fluid for small Rossby numbers, using linearized equations, shows that waves of all modes are excited owing to a sudden flow discharge at a point sink, and propagate at various speeds upstream. The nth mode $A_n J_0(\lambda_n r)$ of the wave travels at a speed $c_0 \lambda_1/\lambda_n$, where λ_n is the nth zero of $J_1(\lambda)$. Thus the first

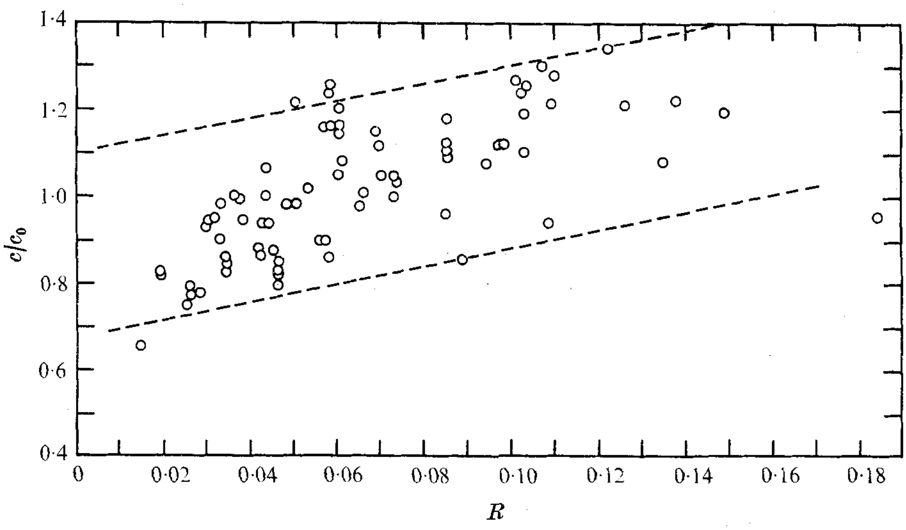

FIGURE 11. Ratio of the blocking wave speed and the maximum group velocity of the infinitesimal waves *vs*. Rossby number.

mode $A_1 J_0(\lambda_1 r)$ of the wave is the fastest disturbance and travels at the maximum wave speed c_0. All higher modes travel at progressively slower speeds. Since the linear theory is expected to hold for small values of R, it is seen from figure 11 that the linear theory is probably valid for $R < 0{\cdot}03$. In this case the waves are dispersive and an ever-lengthening transition region associated with the wave group will develop. Then in the blocking wave model represented by figure 7 an equivalent speed of the blocking wave group has been defined. For example, for $R = 0{\cdot}02$ we have $c \simeq 0{\cdot}8c_0$. That means the wave group is essentially represented by the first few modes of dispersive waves with an equivalent speed of $0{\cdot}8c_0$.

Finally, a word about the energy dissipation in the transition region is in order. The analysis in this section indicates that the angular momentum along the stream tube in figure 7 is essentially conserved while the energy is not. In fact, some energy dissipation is essential so that such a transition is possible. Otherwise, a uniform rotating flow in a straight tube will remain uniform according to equation (2.7).

8. Conclusions

From the present study, we draw the following conclusions.

(i) Based on the experimental observations, a theoretical flow model involving a surface of velocity discontinuity which separates the central flowing core from the surrounding stagnant region is proposed. Indeterminancy arises for various sizes of the flowing core. However, it is found that, whenever selective withdrawal occurs, the flowing core tends to adjust itself such that the flow possesses a minimum flow force as well as a minimum energy flux. Such a flow is called a pseudo-critical blocked flow.

(ii) For the physical flow of selective widthdrawal, a unique intrinsic Rossby number R' based on the properties of the flowing core is determined with a value of $1/\sqrt{8}$, which agrees very well with the experimental value 0·36 obtained in I.

(iii) Based on the experimental observations a theoretical model involving a blocking wave propagating upstream is proposed. It is deduced that for $R > 0{\cdot}06$ the speeds of blocking waves are higher than the maximum group velocity $0{\cdot}522\Omega b$ of the infinitesimal waves. In this case, the blocking waves are non-linear in nature. However, for $R < 0{\cdot}03$, the equivalent speeds of blocking waves are lower than $0{\cdot}522\Omega b$. In this case, the waves are essentially linear and dispersive in nature.

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Appendix

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The method used by Kao (1965*a*, 1970) to construct a dividing streamline in the flow of a stratified fluid in a channel of unit normalized depth (and used in an analogous manner in the present paper) entails the introduction of a sink distribution on the end wall (i.e. on $x = 0$, between $a \leqslant y \leqslant 1$, where x is horizontal and y is vertical) in addition to the two-dimensional point sink at the origin. In order that the dividing streamline may serve as a free streamline on which the normalized speed is unity in the interior of the flow field, the dividing streamline must meet the end wall ($x = 0$) tangentially. A question arises as to whether there are conditions on the sink distribution under which the tangency of the dividing streamline to the end wall is assured.

It will now be shown that the dividing streamline will meet the line $x = 0$ at $y = a$ tangentially if the strength of the sink distribution falls to zero linearly at $y = a$ and if the strength of the point sink relative to that of the sink distribution is sufficiently strong in a sense that will be made explicit later. On the other hand, if the strength of the sink distribution is finite at $y = a$, such as in the case of a uniform sink distribution, there is a local singularity of a logarithmic nature for the vertical velocity component v. In that case the dividing streamline must meet the line $x = 0$ in a stagnation point in the interval $0 < y < a$. It will however also be shown that the logarithmic singularity is felt only in a small neighbourhood of the point $y = a$, so that by letting the uniform sink distribution fall off to zero rapidly in a linear manner in a small interval $a \leqslant y \leqslant a+\epsilon$, where ϵ is positive, the singularity is easily removed and tangency at $y = a$ is assured for a sufficiently strong sink at the origin. Thus for calculational purposes the uniform sink distribution is adequate.

To examine the nature of the velocity at the point $y = a$ one may first consider the following for two-dimensional potential flow in an infinite medium. Let a sink distribution of strength $m(y)$ be located on the line $x = 0$ in

$$a \leqslant y \leqslant 2-a \quad (0 < a < 1).$$

The stream function is

$$\psi = \frac{1}{2\pi}\int_a^{2-a} m(\eta)\tan^{-1}\left(\frac{x}{y-\eta}\right)d\eta,$$

and

$$v = \frac{1}{2\pi}\int_a^{2-a} m(\eta)\frac{y-\eta}{(y-\eta)^2+x^2}d\eta.$$

We shall presently introduce point sinks at $y = 0$ and 2. For this symmetric situation about $y = 1$, we need only consider $0 \leqslant y \leqslant 1$. Now, on $x = 0$, v has a logarithmic singularity at the point $y = a$ if $m(a)$ is finite. On the other hand, if $m(\eta)$ behaves like $\eta - a$ around $y = a$ the value of v is finite and positive there.

In the latter case the horizontal velocity component u is zero for $x = 0$ and $0 < y \leqslant a$. Thus, if a sink is located at the origin and the magnitude of the velocity v in the interval $0 < y \leqslant a$ due to the point sink is greater than the corresponding values due to the sink distribution, the dividing streamline must meet the point $y = a$ tangentially.

For a finite channel of depth normalized to unity, the arguments outlined above remain substantially the same. These may be made more precise as follows. The solution for a uniform sink distribution situated at $x = 0$ and along $a \leqslant y \leqslant 1$ is

$$\psi = -y + \sum_{n=1}^{\infty}\left(\frac{2}{1-a}\right)\frac{\sin n\pi a}{n^2\pi^2}\exp(n\pi x)\sin n\pi y \quad (x \leqslant 0,\ 0 \leqslant y \leqslant 1),$$

with

$$v = \partial\psi/\partial x.$$

The differentiated series converges for all $x < 0$. Direct computation of v at $y = a$ and $|x| = 0{\cdot}01$, $0{\cdot}001$ and $0{\cdot}0001$ reveals the logarithmic singularity there and gives $v \sim |0{\cdot}38\log|x||$ as $x \to 0$, the radius of influence being of $O(10^{-3})$. (For stratified fluid the factor $\exp(n\pi x)$ is replaced by $\exp[(n^2\pi^2 - F^{-2})^{\frac{1}{2}}x]$, where F is a Froude number whose value is greater than $1/\pi$, but the nature of the singularity is the same.) Now if the uniform sink distribution is allowed to decrease its strength linearly from $y = a+\epsilon$ to $y = a$, where ϵ is say of $O(10^{-2})$, the flow field is unchanged for $|x| > 0{\cdot}01$ and v approaches a finite positive number as $x \to 0$ at $y = a$. A direct computation using the appropriate series solution with $a = 0{\cdot}23$, $\epsilon = 0{\cdot}02$ and $\epsilon = 0{\cdot}005$ demonstrated the above result giving $v \simeq 1{\cdot}9$ and $2{\cdot}4$ respectively as $x \to 0$. Indeed by adjusting the value of ϵ, v can be made to take on different values. Therefore for practical calculation the uniform sink distribution is recommended. A more refined calculation for $|x| < 0{\cdot}01$ may subsequently be performed by letting the uniform distribution to go to zero linearly at $y = a$. This refinement of course may only be carried out at considerable expense of computation time since the series solutions converge extremely slowly for small values of x. Its execution is probably unwarranted and may be omitted.

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